CATEGORY-ISOMORPHISMS AND ENDOMORPHISM RINGS OF MODULES

BY

KIITI MORITA

1. Introduction. As is well known, any two faithful modules over a simple ring are isomorphic if their endomorphism rings are isomorphic. This theorem, however, cannot be extended to the case of modules over a semi-simple ring. As a generalization of isomorphisms between modules over a ring we have the notion of semi-linear isomorphisms. But even this notion is not general enough for the above theorem to be generalized to the case of modules over a semi-simple ring([1]). It seems to the author that the most general type of condition which implies the existence of an isomorphism between the endomorphism rings of two modules is to require the existence of a category-isomorphism carrying one of these modules into the other, and by the notion of category-isomorphisms we can obtain a generalization of the above theorem in a satisfactory form even for modules over a quasi-Frobenius ring.

Let A and B be associative rings each of which satisfies the minimum condition for left and right ideals and has a unit element. Let $\mathfrak{A}$ be a class of right (or left) A-modules and $\mathfrak{B}$ a class of right (or left) B-modules. A function T, which assigns to each module X in $\mathfrak{A}$ a B-module $T(X)$ in $\mathfrak{B}$ and to each A-homomorphism $f: X \rightarrow X'(X, X' \in \mathfrak{A})$ a B-homomorphism $T(f): T(X) \rightarrow T(X')$, is called a covariant functor from the category $\mathfrak{A}$ to the category $\mathfrak{B}([2])$ if the following three conditions are satisfied:

(1) If $f: X \rightarrow X$ is the identity then $T(f)$ is the identity;

(2) $\qquad T(f' \circ f) = T(f') \circ T(f) \qquad$ for $f: X \rightarrow X', f': X' \rightarrow X''$;

(3) $\qquad T(f + f') = T(f) + T(f') \qquad$ for $f, f': X \rightarrow X'$.

Let T and T' be two covariant functors from $\mathfrak{A}$ to $\mathfrak{B}$. A natural transformation $\Phi: T \rightarrow T'$ is defined to be a family of B-homomorphisms $\Phi(X): T(X) \rightarrow T'(X)(X \in \mathfrak{A})$ such that $T'(f) \circ \Phi(X) = \Phi(X') \circ T(f)$ for each A-homomorphism $f: X \rightarrow X'$. If each $\Phi(X)$ is a B-isomorphism of $T(X)$ onto $T'(X)$ then $\Phi: T \rightarrow T'$ is called a natural equivalence. If there exists a natural equivalence $\Phi: T \rightarrow T'$, T and T' are said to be naturally equivalent.

---

[1] Let $A = A_1 \oplus A_2$ be a semi-simple ring such that A_1 and A_2 are isomorphic respectively to the full matrix rings $(K)_2$ and $(K)_3$ over a field K. If we denote by L_i the simple right ideal of A_i $(i = 1, 2)$ and put $X = L_1 \oplus L_2 \oplus L_2$, $Y = L_1 \oplus L_1 \oplus L_2$, then the endomorphism rings of X and Y are isomorphic, but there is no semi-linear isomorphism of X onto Y.

[2] The category in which "objects" are modules in $\mathfrak{A}$ and "maps" are all A-homomorphisms between modules in $\mathfrak{A}$ will be denoted by the same letter $\mathfrak{A}$. Here we are considering additive functors. As for functors, cf. Cartan and Eilenberg [5].

Let T_1 be a covariant functor from $\mathfrak{A}$ to $\mathfrak{B}$ and T_2 a covariant functor from $\mathfrak{B}$ to $\mathfrak{A}$. In case the composite functors T_2T_1 and T_1T_2 are naturally equivalent to the identity functor, T_1(resp. T_2) is called a category-isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ (resp. from $\mathfrak{B}$ onto $\mathfrak{A}$)[3].

For a ring C we shall denote always by $\mathfrak{M}_C$ the category of all right C-modules and by $\mathrm{End}_C(M)$ the C-endomorphism ring of a right C-module M.

With these notations we have

PROPOSITION 1.1. *Let X be a right A-module and Y a right B-module. If there exists a category-isomorphism T from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ such that $T(X)$ is B-isomorphic to Y, then $\mathrm{End}_A(X)$ is ring-isomorphic to $\mathrm{End}_B(Y)$.*

Indeed, the correspondence $f \rightarrow \psi \circ T(f) \circ \psi^{-1}$ for $f \in \mathrm{End}_A(X)$ gives a ring-isomorphism of $\mathrm{End}_A(X)$ onto $\mathrm{End}_B(Y)$ where ψ is a B-isomorphism of $T(X)$ onto Y.

The present paper is concerned with the problem: For what kind of rings A and B does the converse of Proposition 1.1 hold?

In discussing this problem we shall first restrict ourselves to the case of faithful modules. It can be shown that if A and B are quasi-Frobenius rings and if a right A-module X and a right B-module Y are faithful modules which are direct sums of finitely generated submodules then the converse of Proposition 1.1 holds. However, we can establish a more general and more precise result.

We shall say that a subring C of $\mathrm{End}_A(X)$ is *strongly dense* in $\mathrm{End}_A(X)$ if the following two conditions are satisfied:

(1) C is dense in $\mathrm{End}_A(X)$ in the ordinary sense; that is, for any finite number of elements $x_1, \cdots, x_m$ in X and for any endomorphism $f \in \mathrm{End}_A(X)$ there exists $g \in C$ such that $g(x_i) = f(x_i)$ for $i = 1, 2, \cdots, m$.

(2) For each finitely generated A-submodule M of X which is a direct summand of X there exists $g \in C$ such that g is a projection of X onto M.

Here we shall recall some definitions. Let A be a ring which has a unit element and satisfies the minimum condition for left and right ideals. Then A is called quasi-Frobenius if $\mathrm{Hom}_A(\mathrm{Hom}_A(X, A), A) \cong X$ holds for every finitely generated left or right A-module X where $\mathrm{Hom}_A(Y, A)$ for a left (resp. right) A-module Y means the right (resp. left) A-module which consists of all A-homomorphisms of Y into A with the usual compositions: $(\alpha + \beta)(y) = \alpha(y) + \beta(y)$, $(\alpha a)(y) = \alpha(y)a$ (resp. $(a\alpha)(y) = a(\alpha(y))$), for $\alpha, \beta \in \mathrm{Hom}_A(Y, A)$, $a \in A$, $y \in Y$. A is called generalized uni-serial if every left ideal Ae as well as every right ideal eA generated by a primitive idempotent element e possesses only one composition series. A is called uni-serial if A is generalized uni-serial and if A is a direct sum of two-sided ideals each of which is a primary ring. Semi-simple rings are uni-serial, and uni-serial rings

---

are quasi-Frobenius, and group rings of finite groups over a field are also quasi-Frobenius.

Now we can state our main theorem.

THEOREM 1.2. *Let A and B be quasi-Frobenius rings. Let X and Y be respectively a faithful right A-module and a faithful right B-module such that X and Y are direct sums of finitely generated submodules. Let C be a strongly dense subring of $\mathrm{End}_A(X)$ and D a strongly dense subring of $\mathrm{End}_B(Y)$. If Φ is a ring-isomorphism of C onto D, then there exists a category-isomorphism T from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ such that $T(X)$ is B-isomorphic to Y and*

$$\Phi(f) = \psi \circ T(f) \circ \psi^{-1} \qquad\qquad for\ f \in C$$

holds with a B-isomorphism ψ of $T(X)$ onto Y.

As a special case where $C = \mathrm{End}_A(X)$ and $D = \mathrm{End}_B(Y)$ Theorem 1.2 contains a solution to the problem proposed above in its precise form.

In applications of Theorem 1.2 it will be useful to note that any module over a generalized uni-serial ring is a direct sum of cyclic submodules (cf. Nakayama [11]) and hence a direct sum of finitely generated submodules.

In case A and B are self-basic, that is, each of them coincides with its basic ring, Theorem 1.2 can be brought into another form:

THEOREM 1.3. *Let A and B be self-basic quasi-Frobenius rings. Let X, Y, C, D be as in Theorem 1.2. Suppose that Φ is a ring-isomorphism of C onto D. Then there exists a semi-linear isomorphism ω of X onto Y with a ring-isomorphism θ of A onto B[4] such that*

$$\Phi(f) = \omega \circ f \circ \omega^{-1}, \qquad\qquad for\ f \in C.$$

As is well known, any dense subring of the ring of all linear transformations of a vector space over a division ring is strongly dense in our sense if it contains a nonzero linear transformation of finite rank. Accordingly it is seen that the isomorphism theorem for primitive rings with minimal left ideals (cf. Jacobson [7]) may be obtained as a special case of Theorem 1.3; we have only to notice that a division ring is self-basic, generalized uni-serial and quasi-Frobenius. Likewise Shoda's theorem [14], Asano's theorem [1], and Baer's theorem [4], which are concerned with endomorphism rings of modules over a commutative or completely primary uni-serial ring, are also special cases of our Theorem 1.3.

In the above theorems we have dealt with the case of faithful modules. In case we admit X and Y to be nonfaithful modules, we must restrict ourselves to the case where A and B are similar[5]. This restriction is justified by the

---

[4] That is, ω is a one-to-one mapping of X onto Y such that $\omega(x+x')=\omega(x)+\omega(x')$, $\omega(xa)=\omega(x)\theta(a)$ for x, $x'\in X$, $a\in A$.

[5] If the basic rings of A and B are isomorphic, we shall say that A and B are similar.

following two facts: (1) if there exists a category-isomorphism from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ then A and B are similar (cf. Morita [8, Theorem 7.5]), and (2) for a nonzero two-sided ideal I of A the residue class ring A/I is not similar to A but $\mathrm{End}_A(X) \cong \mathrm{End}_{A/I}(X)$ for any right A/I-module X. Our results in this case will be stated in §7.

The types of rings to which A and B are restricted in the above theorems are shown (under some condition) to be general enough for the conclusions in these theorems to hold. From this point of view we have a new characterization of quasi-Frobenius rings (§6).

If in Theorems 1.2 and 1.3 we impose on X and Y a further restriction that X and Y should be fully faithful in the sense defined in §3 then these theorems remain true for any rings A and B with a slight condition which is satisfied for any algebras over a commutative field; these results are stated in §3 as Theorems 3.2 and 3.3 from which Theorems 1.2 and 1.3 will be deduced as immediate corollaries.

Throughout this paper the meanings of the notations used in this introduction will be retained; thus, A and B are always assumed to be associative rings each of which has a unit element and satisfies the minimum condition for left and right ideals. Modules over a ring are assumed to be unital in the sense that the unit element of the ring acts on the modules as the identity operator.

2. **Category-isomorphisms.** Let us denote by 1 the unit element of A. Then there exist mutually orthogonal primitive idempotent elements $e_{k,i}$, $k = 1, \cdots, m; i = 1, \cdots, r(k)$ such that

$$1 = \sum_{k=1}^{m} \sum_{i=1}^{r(k)} e_{k,i}; \qquad A e_{k,i} \cong A e_{l,j} \qquad \text{if and only if } k = e.$$

Let us put

$$A^0 = eAe, \qquad e = \sum_{k=1}^{m} e_{k,1}.$$

A^0 is called the basic ring of A; the basic ring of A is determined by A uniquely up to an inner automorphism (cf. Osima [13]).

Let us set

$$P_1(X) = X \otimes_A Ae, \qquad \text{for } X \in \mathfrak{M}_A,$$
$$P_2(X^0) = \mathrm{Hom}_{A^0}(Ae, X^0), \qquad \text{for } X^0 \in \mathfrak{M}_{A^0},$$

where Ae is considered as a two-sided $A - A^0$-module. Then, as is shown in Morita [8, §7], P_1 is a category-isomorphism from $\mathfrak{M}_A$ onto $\mathfrak{M}_{A^0}$ and P_2 a category-isomorphism from $\mathfrak{M}_{A^0}$ onto $\mathfrak{M}_A$, and the composite functors $P_2 P_1$ and $P_1 P_2$ are naturally equivalent to the identity functor.

For B we shall construct the corresponding functors:

$$Q_1(Y) = Y \otimes_B Be' \qquad\qquad \text{for } Y \in \mathfrak{M}_B,$$

$$Q_2(Y^0) = \mathrm{Hom}_{B^0}(Be', Y^0), \qquad\qquad \text{for } Y^0 \in \mathfrak{M}_{B^0}$$

where e' is defined similarly as e for A, $B = e'Be'$ is the basic ring of B, and Be' is considered as a two-sided $B - B^0$-module. The composite functors Q_2Q_1 and Q_1Q_2 are naturally equivalent to the identity functor, and $Q_1: \mathfrak{M}_B \to \mathfrak{M}_{B^0}$, $Q_2: \mathfrak{M}_{B^0} \to \mathfrak{M}_B$ are category-isomorphisms.

Let T be any category-isomorphism from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$. Then

$$T^0 = Q_1 T P_2: \mathfrak{M}_{A^0} \to \mathfrak{M}_{B^0}$$

is a category-isomorphism and T is naturally equivalent to a category-isomorphism $Q_2 T^0 P_1$. Thus the diagram

$$
\begin{array}{ccc}
\mathfrak{M}_A & \xrightarrow{\ \ T\ \ } & \mathfrak{M}_B \\[4pt]
P_2 \Updownarrow P_1 & & Q_1 \Updownarrow Q_2 \\[4pt]
\mathfrak{M}_{A^0} & \xrightarrow[\ \ T^0\ \ ]{} & \mathfrak{M}_{B^0}
\end{array}
$$

is commutative in the sense that the composite functors indicated by arrows are naturally equivalent if they are defined over the same category and have values in the other same category.

Thus the study of category-isomorphisms from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ is reduced to that of category-isomorphisms from $\mathfrak{M}_{A^0}$ onto $\mathfrak{M}_{B^0}$. As for the latter the following theorem is fundamental.

THEOREM 2.1. *Let $\mathfrak{A}$ be a class of right A-modules containing A as a right A-module and $\mathfrak{B}$ a class of right B-modules containing B as a right B-module. Let T_1 and T_2 be category-isomorphisms from $\mathfrak{A}$ onto $\mathfrak{B}$ and from $\mathfrak{B}$ onto $\mathfrak{A}$ such that $T_2 T_1$ and $T_1 T_2$ are naturally equivalent to the identity functor. Suppose that there is an A-isomorphism $\sigma: A \to T_2(B)$. Then there exists a ring-isomorphism θ from A onto B such that*

$$(4) \qquad\qquad T_2(\psi_{\theta(a)}) = \sigma \circ \phi_a \circ \sigma^{-1}, \qquad\qquad \text{for } a \in A$$

holds. (1) is equivalent to

$$(5) \qquad\qquad \sigma(au) = \theta(a)\sigma(u), \qquad\qquad \text{for } u \in A,\, a \in A.$$

Here $\phi_a: A \to A$ is a right A-homomorphism defined by $\phi_a(x) = ax$ for $x \in A$ and $\psi_b: B \to B$ a right B-homomorphism defined by $\psi_b(y) = by$ for $y \in B$, and bv is defined to be $T_2(\psi_b)v$ for $v \in T_2(B)$, $b \in B$.

In this case $V = T_2(B)$ becomes a two-sided $B - A$-module and T_1 is naturally equivalent to the functor $\mathrm{Hom}_A(V,\ )$. Furthermore

$$(6) \qquad\qquad \Gamma(X): X \to \mathrm{Hom}_A(V, X) \qquad\qquad \text{for } X \in \mathfrak{A}$$

defined by $\Gamma(X)(x) = \phi_x \circ \sigma^{-1}$ is a semi-linear isomorphism with a ring-isomor-

phism θ of A onto B, that is, $\Gamma(X)(xa) = [\Gamma(X)(x)]\theta(a)$, where $\phi_x: A \to X$ is defined by $\phi_x(a) = xa$ for $x \in X$, $a \in A$.

Proof. Since $\sigma^{-1} \circ T_2(\psi_b) \circ \sigma$ is an A-endomorphism of the right A-module A, there exists uniquely an element of A, which will be denoted by $\chi(b)$, such that $\sigma^{-1} \circ T_2(\psi_b) \circ \sigma = \phi_{\chi(b)}$. Then χ is easily shown to be a ring-isomorphism of B onto A. If we put $\theta = \chi^{-1}$, we have the first part of the theorem.

The second part is proved in Morita [8, Theorem 3.1]. As for the last part we have

$$\begin{aligned}
[\Gamma(X)(xa)](v) &= (\phi_{xa} \circ \sigma^{-1})(v) = (xa)[\sigma^{-1}(v)] \\
&= x[a(\sigma^{-1}(v))] = x[\sigma^{-1}(\theta(a)v)] = (\phi_x \circ \sigma^{-1})(\theta(a)v) \\
&= [\Gamma(X)(x)](\theta(a)v) = [(\Gamma(X)(x))\theta(a)](v).
\end{aligned}$$

This completes the proof.

In Theorem 2.1, if we put $v_0 = \sigma(1)$, then we have

(7) $V = \{v_0 a \mid a \in A\}$, $v_0 a = 0$ if and only if $a = 0$,

(8) $v_0 a = \theta(a) v_0$ for $a \in A$.

Conversely, for a given ring-isomorphism θ of A onto B we can construct a two-sided $B - A$-module V by the formulas (7) and (8). Then $\sigma: A \to V$ defined by $\sigma(a) = v_0 a$ is an A-isomorphism satisfying (5), and the covariant functor T defined by

$$T(X) = \mathrm{Hom}_A(V, X), \qquad \text{for } X \in \mathfrak{M}_A,$$

$$[T(f)](\alpha) = f \circ \alpha, \qquad \text{for } \alpha \in \mathrm{Hom}_A(V, X), f \in \mathrm{Hom}_A(X, X')$$

is a category-isomorphism from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ by Morita [8, Theorem 3.4] and the ring-isomorphism associated with T by Theorem 2.1 is equal to θ.

LEMMA 2.2. *With respect to the category-isomorphism T defined above we have*

$$T(f) = \Gamma(X) \circ f \circ \Gamma(X)^{-1}, \qquad \text{for } f \in \mathrm{End}_A(X),$$

where $X \in \mathfrak{M}_A$ and $\Gamma(X)$ is defined by (6).

Proof. For the sake of brevity we set $\omega_1 = \Gamma(X)$ and $S(f) = \Gamma(X) \circ f \circ \Gamma(X)^{-1}$. Let $\alpha \in \mathrm{Hom}_A(V, X)$, $a \in A$, and put $x = \alpha(v_0)$. Then we have

$$[\omega_1(x)](v_0 a) = (\phi_x \circ \sigma^{-1})(v_0 a) = \phi_x(\sigma^{-1}(v_0 a)) = \phi_x(a) = xa,$$

$$\alpha(v_0 a) = [\alpha(v_0)]a = xa,$$

and hence $\omega_1(x) = \alpha$. Therefore

$$S(f)(\alpha) = (\omega_1 \circ f \circ \omega_1^{-1})(\alpha) = \omega_1(f(x))$$

and consequently

$$[S(f)(\alpha)](v_0 a) = [\omega_1(f(x))](v_0 a) = f(x)a.$$

On the other hand, we have

$$[T(f)(\alpha)](v_0 a) = (f \circ \alpha)(v_0 a) = f(\alpha(v_0 a)) = f(xa) = f(x)a.$$

This completes our proof.

LEMMA 2.3. *Let ω be a semi-linear isomorphism of a right A-module X onto a right B-module Y with a ring-isomorphism θ of A onto B. If we put $\psi = \omega \circ \Gamma(X)^{-1}$, then ψ is a B-isomorphism of $T(X)$ onto Y and*

$$\omega \circ f \circ \omega^{-1} = \psi \circ T(f) \circ \psi^{-1} \qquad \textit{for } f \in \mathrm{End}_A(X).$$

Proof. This is an immediate consequence of Lemma 2.2.

Now let A and B be self-basic rings. Then for any category-isomorphism T_2 from $\mathfrak{M}_B$ onto $\mathfrak{M}_A$ there exists always an A-isomorphism $\sigma: A \to T_2(B)$; this is seen from the fact that the projectivity and indecomposability of modules are preserved under any category-isomorphism (cf. Morita [8, §13]). Therefore any category-isomorphism from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ is determined by a ring-isomorphism θ of A onto B.

3. Main theorem. In this section we shall state our main theorem in a more general setting.

We shall say that a right A-module X is *fully faithful*, if every indecomposable projective right A-module as well as every indecomposable injective right A-module is A-isomorphic to a direct summand of X. Then we have clearly (e.g., cf. [8, Theorems 14.1 and 16.5]).

PROPOSITION 3.1. *A is a quasi-Frobenius ring if and only if every faithful right A-module is fully faithful.*

We are now in a position to state our main theorem in a more general form.

THEOREM 3.2. *Assume that every indecomposable injective right A-module and every indecomposable injective right B-module are finitely generated. Let X be a fully faithful right A-module and Y a fully faithful right B-module, and let X and Y be direct sums of finitely generated submodules. Let C be a strongly dense subring of $\mathrm{End}_A(X)$ and D a strongly dense subring of $\mathrm{End}_B(Y)$. If Φ is a ring-isomorphism of C onto D, then there exists a category-isomorphism T from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ such that $T(X)$ is B-isomorphic to Y and*

$$\Phi(f) = \psi \circ T(f) \circ \psi^{-1} \qquad \textit{for } f \in C$$

holds with a B-isomorphism ψ of $T(X)$ onto Y.

The following theorem corresponds to Theorem 1.3.

THEOREM 3.3. *Let A, B, X, Y, C, and D be as in Theorem 3.2. Assume further that A and B are self-basic. If Φ is a ring-isomorphism of C onto D, then*

there exists a semi-linear isomorphism ω of X onto Y with a ring-isomorphism θ of A onto B such that

$$\Phi(f) = \omega \circ f \circ \omega^{-1}, \qquad\qquad for\ f \in C.$$

From Proposition 3.1 it follows that Theorems 1.2 and 1.3 are special cases of Theorems 3.2 and 3.3 respectively. We shall prove here

PROPOSITION 3.4. *Theorem 3.2 can be deduced from Theorem 3.3.*

Let A, B, X, Y, C and D be the same as in Theorem 3.2, and let Φ be a ring-isomorphism of C onto D. With the notations used at the beginning of §2 we set

$$X^0 = P_1(X), \qquad Y^0 = Q_1(Y),$$
$$C^0 = \{P_1(f) \,|\, f \in C\}, \qquad D^0 = \{Q_1(g) \,|\, g \in D\},$$

$$(9) \qquad\qquad \Phi_0(P_1(f)) = Q_1(\Phi(f)) \qquad\qquad for\ f \in C.$$

Then X^0 and Y^0 are fully faithful modules over A^0 and B^0 respectively, and they are direct sums of finitely generated submodules. C^0 and D^0 are strongly dense subrings of $\mathrm{End}_{A^0}(X^0)$ and $\mathrm{End}_{B^0}(Y^0)$ respectively. Φ_0 is a ring-isomorphism of C^0 onto D^0. Moreover, A^0 and B^0 are self-basic rings such that every indecomposable injective right module over A^0 (resp. B^0) is finitely generated.

Let us now assume that Theorem 3.3 is true. Then there exists a semi-linear isomorphism ω_0 of X^0 onto Y^0 such that $\Phi_0(P_1(f)) = \omega_0 \circ P_1(f) \circ \omega_0^{-1}$ for $f \in C$. Hence by Lemma 2.3 there exists a category-isomorphism T_0 from $\mathfrak{M}_{A^0}$ onto $\mathfrak{M}_{B^0}$ such that

$$(10) \qquad\qquad \Phi_0(P_1(f)) = \psi_0 \circ T_0(P_1(f)) \circ \psi_0^{-1} \qquad\qquad for\ f \in C$$

holds with a B^0-isomorphism $\psi_0 \colon T_0(X^0) \cong Y^0$.

From (9) and (10) we get

$$(11) \qquad (Q_2 Q_1)(\Phi(f)) = Q_2(\psi_0) \circ (Q_2 T_0 P_1)(f) \circ Q_2(\psi_0)^{-1}.$$

Now let μ be the natural equivalence from the identity functor to $Q_2 Q_1$. Then we have

$$(Q_2 Q_1)(\Phi(f)) = \mu(Y) \circ \Phi(f) \circ \mu(Y)^{-1}.$$

Hence, if we put

$$T = Q_2 T_0 P_1, \qquad \psi = \mu(Y)^{-1} \circ Q_2(\psi_0),$$

then T is a category-isomorphism from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$, ψ is a B-isomorphism of $T(X)$ onto Y, and

$$\Phi(f) = \psi \circ T(f) \circ \psi^{-1}, \qquad\qquad \text{for } f \in C.$$

Thus we have proved that Theorem 3.3 implies Theorem 3.2.

4. Some lemmas.

LEMMA 4.1. *Let X be a faithful right A-module which is decomposed into a direct sum of two submodules X_0 and X_1 such that X_0 is A-isomorphic to A as a right A-module. Let C be any dense subring of $\mathrm{End}_A(X)$. Suppose that C contains a projection e_0 from X onto X_0. Then the left C-module X is C-isomorphic to Ce_0, and any C-endomorphism of X is obtained by the right multiplication of an element of A.*

Proof. We assume that $X_1 = (1-e_0)X$. Let x_0 be the element of X_0 corresponding to the unit element of A by the given isomorphism of X_0 onto A. We put $\alpha(x_0 a) = a$ for $a \in A$; $\alpha(x) = 0$ for $x \in X_1$. Then we have $\alpha \in \mathrm{Hom}_A(X, A)$. Then for $x' \in X$ the correspondence $x \to x'\alpha(x)$ defines an A-endomorphism of X which will be denoted by $\sigma(x')$; we have

$$x'[\alpha(x)] = \sigma(x')x, \qquad \sigma(x') \in \mathrm{End}_A(X).$$

For σ we have $\sigma(x+x') = \sigma(x)+\sigma(x')$, $\sigma(cx) = c\sigma(x)$ for $c \in \mathrm{End}_A(X)$. Moreover, we have $\sigma(x_0) = e_0$, $\sigma(x)e_0 = \sigma(x)$ for $x \in X$, since

(1) $$\sigma(x_0)(x_0 a) = x_0\alpha(x_0 a) = x_0 a \qquad\qquad \text{for } a \in A,$$

and

$$\sigma(x_0)(x) = x_0\alpha(x) = 0 \qquad\qquad \text{for } x \in X_1;$$

(2) $\quad \sigma(x)x' = x[\alpha(x')] = x(\alpha(e_0 x')) = \sigma(x)[e_0 x'] = (\sigma(x)e_0)(x') \quad$ for any $x' \in X$.

Thus σ is an $\mathrm{End}_A(X)$-homomorphism of X into $\mathrm{End}_A(X)e_0$. Furthermore, σ is an $\mathrm{End}_A(X)$-isomorphism; (1) for $x \neq x'$ we have $\sigma(x) \neq \sigma(x')$ because $x = x\alpha(x_0) = \sigma(x)x_0$; (2) for any $f \in \mathrm{End}_A(X)$ we have $\sigma(fx_0) = f\sigma(x_0) = fe_0$ and hence σ is onto.

Now let $f \in \mathrm{End}_A(X)$. Then, since C is dense in $\mathrm{End}_A(X)$, there exists $g \in C$ such that $g(x_0) = f(x_0)$. For this g we have $ge_0 = g\sigma(x_0) = \sigma(gx_0) = \sigma(fx_0) = f\sigma(x_0) = fe_0$. Hence $Ce_0 = [\mathrm{End}_A(X)]e_0$. This proves the first part.

To prove the second part, let λ be any C-endomorphism of X. If we put $a = \alpha(\lambda(x_0))$, then for any $x \in X$ we have $xa = x(\alpha(\lambda(x_0))) = \sigma(x)\lambda(x_0) = \lambda(\sigma(x)x_0) = \lambda(x\alpha(x_0)) = \lambda(x)$. (Cf. Morita [8, §3]).

LEMMA 4.2. *Let X be a right A-module which is decomposed into a direct sum of finitely generated A-submodules. Let C be a strongly dense subring of $\mathrm{End}_A(X)$. If e is a primitive idempotent element of C, then e is primitive also in $\mathrm{End}_A(X)$ and eX is a finitely generated, indecomposable A-module.*

Proof. From the assumption it follows that X is a direct sum of indecomposable, finitely generated A-submodules $X_\lambda (\lambda \in \Lambda)$: $X = \Sigma \oplus X_\lambda$. Then for each λ there exists an idempotent element $e_\lambda \in C$ such that $X_\lambda = e_\lambda X$. By

Azumaya [2, Theorem 1] there exist some e_λ and an idempotent element $f \in C$ such that e is one-to-one on $e_\lambda X$, and $ee_\lambda X = fX$ and f is primitive in $\text{End}_A(X)$. Hence we have

$$ef = f, \qquad fee_\lambda = ee_\lambda.$$

Furthermore, we have

$$e = efe + (e - efe)$$
$$(efe)(efe) = efe, \qquad efe(e - efe) = (e - efe)efe = 0.$$

Since $e \in C$, $f \in C$ and e is primitive in C, we have $e = efe$; otherwise we would have $efe = 0$ and hence $ee_\lambda = (efe)e_\lambda = 0$ which contradicts $ee_\lambda X = fX \neq 0$. Hence $fe = (ef)e = e$. Thus we get $ef = f$, $fe = e$. This shows that $eX = fX$. Therefore eX is a finitely generated, indecomposable A-submodule and e is primitive in $\text{End}_A(X)$.

LEMMA 4.3. *Let X and C be as in Lemma 4.2. Let e be an idempotent element of C such that eX is finitely generated. Then e is a sum of a finite number of mutually orthogonal primitive idempotent elements of C.*

Proof. Let $eX = X_1 \oplus \cdots \oplus X_s$ be a decomposition of eX into a direct sum of indecomposable, finitely generated submodules. We shall denote by e_i the A-endomorphism of X which is identity on X_i and zero on $(1-e)X \oplus \sum_{j \neq i} \oplus X_j$. Then we have $e = e_1 + \cdots + e_s$. Since eX is finitely generated and C is dense in $\text{End}_A(X)$, there exist $h_i \in C$ such that $h_i x = e_i x$ for all $x \in eX$. Then we have $e_i x = h_i ex$ for each $x \in X$ and hence $e_i = h_i e$. Since $e, h_i \in C$, we have $e_i \in C$.

5. **Proof of the main theorem.** As was shown in §3, we have only to prove Theorem 3.3.

For this purpose, let A and B be self-basic rings such that every indecomposable injective right A-module (resp. B-module) is finitely generated. Let X (resp. Y) be a fully faithful right A-module (resp. B-module) which is a direct sum of finitely generated submodules. Let C be a strongly dense subring of $\text{End}_A(X)$ and D a strongly dense subring of $\text{End}_B(Y)$. Furthermore, let Φ be a ring-isomorphism of C onto D.

We shall first deal with the case where X and Y are finitely generated.

LEMMA 5.1. *Theorem 3.3 holds if X and Y are finitely generated.*

Proof. Since C and D are dense in $\text{End}_A(X)$ and $\text{End}_B(Y)$ respectively and X and Y are finitely generated, we have $C = \text{End}_A(X)$ and $D = \text{End}_B(Y)$. Since X and Y are fully faithful and finitely generated, by Morita [8, Theorems 16.3 and 16.4] the left C-module X as well as the left D-module Y is finitely generated, faithful, projective and injective[6].

─────────────

[6] In case X is not finitely generated, the left C-module X is still projective but not injective.

Let us set

$$c * y = \Phi(c)y \qquad\qquad \text{for } y \in Y, c \in C.$$

Then Y is a left C-module which is finitely generated, faithful, projective and injective. Since the C-endomorphism ring of X (resp. the D-endomorphism ring of Y) is inverse-isomorphic to A (resp. B) and A (resp. B) is self-basic, the left C-modules X and Y are C-isomorphic to Ce_1 and Ce_2 where each of e_1 and e_2 is a sum of a finite number of mutually orthogonal primitive idempotent elements of C such that any two of them generate nonisomorphic left ideals of C. Here it is to be noted that the ring C is semi-primary in the sense that its radical is nilpotent and the residue class ring modulo its radical satisfies the minimum condition for left and right ideals. Hence the semi-simple part ($=$ socle) of an indecomposable injective left C-module is simple and two injective left C-modules are C-isomorphic if and only if their semi-simple parts are C-isomorphic[7]. In the present situation, each of the semi-simple parts of Ce_1 and Ce_2 is a direct sum of a finite number of mutually nonisomorphic simple left C-modules which exhaust all types of simple left C-modules contained in C. Therefore the semi-simple parts of Ce_1 and Ce_2 are C-isomorphic, and consequently Ce_1 and Ce_2 are C-isomorphic[8]. Let ω be a C-isomorphism of X onto Y. Then we have

$$\omega(cx) = c * \omega(x) = \Phi(c)\omega(x) \qquad\qquad \text{for } x \in X, c \in C.$$

On the other hand, for an element a of A the mapping $y \rightarrow \omega(\omega^{-1}(y)a)$ defines a D-endomorphism of Y and hence there exists an element $\theta(a)$ of B such that $\omega(\omega^{-1}(y)a) = y\theta(a)$. Thus we have $\omega(xa) = \omega(x)\theta(a)$. It is easy to see that θ is a ring-isomorphism of A onto B. This completes the proof of Lemma 5.1.

We shall now return to the general case where X and Y are not necessarily finitely generated.

Let e and e' be idempotent elements of C such that eX is A-isomorphic to the right A-module A and $e'X$ is a direct sum of a finite number of mutually nonisomorphic indecomposable injective right A-modules which exhaust all types of indecomposable injective right A-modules; the existence of e and e' is assured by the assumption that C is strongly dense and X is fully faithful. Let f and f' be idempotent elements of D having the properties corresponding to e and e' respectively.

Then by Lemma 4.3 each of f and f' is a direct sum of a finite number of mutually orthogonal primitive idempotent elements of D, and hence each of

---

(7) Theorem 6.1 in Morita [8] holds for semi-primary rings in the above sense (with the exception of 3⁰).

(8) From the fact that Ce_i is a direct sum of a finite number of mutually nonisomorphic indecomposable injective left ideals of C, it can be shown that Ce_i is a minimal faithful left C-module in the sense that any faithful left C-module has a direct summand which is C-isomorphic to Ce_i ($i = 1, 2$). From this we have another proof for the proposition that Ce_1 and Ce_2 are C-isomorphic.

$\Phi^{-1}(f)$ and $\Phi^{-1}(f')$ is a direct sum of a finite number of mutually orthogonal primitive idempotent elements of C. Hence by Lemma 4.2 we see that $\Phi^{-1}(f)X$ and $\Phi^{-1}(f')X$ are finitely generated right A-modules.

From the assumption that X is decomposed into a direct sum of finitely generated A-submodules $X_\lambda(\lambda\in\Lambda)$, it follows that there exist a finite number of elements λ_j ($j=1,\cdots,s$) of Λ such that eX, $e'X$, $\Phi^{-1}(f)X$ and $\Phi^{-1}(f')X$ are all contained in $\sum_{j=1}^{s}\oplus X_{\lambda_j}$. Let h be an element of C which is a projection of X onto $\sum_{j=1}^{s}\oplus X_{\lambda_j}$. Then by Lemma 4.3 h is an idempotent element which is a sum of a finite number of mutually orthogonal primitive idempotent elements of C. Hence by Lemma 4.2 $\Phi(h)Y$ is a finitely generated right B-module.

The direct summands eX and $e'X$ of X are both contained in hX, and hence they are also direct summands of hX. Hence hX is a finitely generated, fully faithful right A-module.

On the other hand, we have $\Phi^{-1}(f)=h\Phi^{-1}(f)$, $\Phi^{-1}(f')=h\Phi^{-1}(f')$ since $\Phi^{-1}(f)X$ and $\Phi^{-1}(f')$ are contained in hX. Hence $f=\Phi(h)f$, $f'=\Phi(h)f'$. Therefore fY and $f'Y$ are contained in $\Phi(h)Y$, and, since they are direct summands of Y, they also direct summands of $\Phi(h)Y$. This shows that $\Phi(h)Y$ is a finitely generated, fully faithful right B-module.

Thus we have arrived at the following situation: hX is a finitely generated, fully faithful right A-module, $\Phi(h)Y$ is a finitely generated, fully faithful right B-module, Φ induces a ring-isomorphism of hCh onto $\Phi(h)D\Phi(h)$, and $hCh = \mathrm{End}_A(hX)$, $\Phi(h)D\Phi(h) = \mathrm{End}_B(\Phi(h)Y)$.

We now apply Lemma 5.1 to the present situation. Then there exists a semi-linear isomorphism ω_0 of hX onto $\Phi(h)Y$ with a ring-isomorphism θ_0 of A onto B such that

$$\omega_0(cx) = \Phi(c)\omega_0(x), \qquad \text{for } x \in hX,\ c \in hCh,$$

$$\omega_0(xa) = \omega_0(x)\theta_0(a), \qquad \text{for } x \in hX,\ a \in A.$$

Since eX is A-isomorphic to the right A-module A, there exists an element x_0 of eX such that $eX = x_0A$ and $x_0a = 0$ for $a \in A$ if and only if $a = 0$. If we put $y_0 = \omega_0(x_0)$, then we have

$$\omega_0(eX) = \omega_0(hehX) = \Phi(h)\Phi(e)\Phi(h)Y = \Phi(e)Y,$$

$$\omega_0(eX) = \omega_0(x_0A) = \omega_0(x_0)\theta_0(A) = y_0B,$$

and $y_0b = 0$ for $b \in B$ if and only if $b = 0$. Thus we have proved that $\Phi(e)Y$ is B-isomorphic to the right B-module B. Therefore by Lemma 4.1 Y is D-isomorphic to $D\Phi(e)$ as left D-modules. On the other hand, again by Lemma 4.1 X is C-isomorphic to Ce as left C-modules.

Of course, Φ induces a semi-linear isomorphism of the left C-module Ce onto the left D-module $D\Phi(e)$ by the correspondence $c\to\Phi(c)$ for $c\in Ce$.

Therefore there exists a semi-linear isomorphism ω of the left C-module X onto the left D-module Y with the ring-isomorphism Φ of C onto D:

$$\omega(cx) = \Phi(c)\omega(x), \qquad \text{for } x \in X, c \in C.$$

For an element a of A, the mapping $y \to \omega(\omega^{-1}(y)a)$ defines a D-endomorphism of Y and hence by Lemma 4.1 there exists an element $\theta(a)$ of B such that $\omega(\omega^{-1}(y)a) = y\theta(a)$. It is easy to see that θ is a ring-isomorphism of A onto B. Thus we have

$$\omega(xa) = \omega(x)\theta(a) \qquad \text{for } x \in X, a \in A,$$

and ω is a semi-linear isomorphism of the right A-module X onto the right B-module Y with a ring-isomorphism θ of A onto B. This completes our proof of Theorem 3.3.

REMARK. In Theorems 3.2 and 3.3 the adjective "fully faithful" cannot be replaced by "faithful" in the usual sense.

Let C be the subalgebra of the full matrix ring $(K)_{10}$ over a commutative field K such that the elements

$$e_1 = c_{11}, \qquad e_2 = c_{22} + c_{55}, \qquad e_3 = c_{33} + c_{88},$$

$$e_4 = c_{44} + c_{66} + c_{99}, \qquad e_5 = c_{77}, \qquad e_6 = c_{10,10},$$

$$c_{21}, \quad c_{31}, \quad c_{41}, \quad c_{42} + c_{65}, \quad c_{43} + c_{98},$$

$$c_{75}, \quad c_{76}, \quad c_{10,8}, \quad c_{10,9},$$

form a K-basis of C, where c_{ik} mean matrix units. The algebra is self-basic, and QF-3 and QF-1 in the sense of Thrall (cf. Morita [9, Example 2.2]). Let us put

$$\begin{aligned}
X &= Ce, & A &= eCe, & e &= e_1 + e_2 + e_3 \\
Y' &= Ce', & B' &= e'Ce', & e' &= e + e_5 \\
Y'' &= Ce'', & B'' &= e''Ce'', & e'' &= e + e_6.
\end{aligned}$$

Then X is a fully faithful right A-module, and Y' (resp. Y'') is a faithful right B'-module (resp. B''-module) which has a direct summand B'-isomorphic to B' (resp. B''-isomorphic to B''). Since C is QF-1, we have $\text{End}_A(X) \cong C$, $\text{End}_{B'}(Y') \cong C$, and $\text{End}_{B''}(Y'') \cong C$. Hence $\text{End}_A(X) \cong \text{End}_{B'}(Y')$ but there exists no semi-linear isomorphism of X onto Y since A and B' are not ring-isomorphic. On the other hand, if we correspond the B''-endomorphism of Y'': $y'' \to cy''(y'' \in Y'')$ to the B'-endomorphism of Y': $y' \to cy'(y' \in Y')$, we have a ring-isomorphism Φ of $\text{End}_{B'}(Y')$ onto $\text{End}_{B''}(Y'')$ but there exists no semi-linear isomorphism ω of Y' onto Y'' such that $\Phi(f) = \omega \circ f \circ \omega^{-1}$ for each $f \in \text{End}_{B'}(Y')$ since such an ω would give a C-isomorphism of Y' onto Y''. (However, there is a semi-linear isomorphism of Y' onto Y'' which is induced by a ring-automorphism of C.)

6. **A characterization of quasi-Frobenius rings.** In this section we shall show under some condition that quasi-Frobenius rings are the most general type of rings for which the conclusion described in Theorem 1.2 holds.

We shall say that A has a duality for modules if there exists a duality between the category $_A\mathfrak{M}^*$ of all finitely generated left A-modules and the category $\mathfrak{M}_A^*$ of all finitely generated right A-modules([9]).

It is well known that an algebra of finite rank over a commutative field or a commutative ring with minimum condition has a duality for modules. It is an open question whether any ring with minimum condition has a duality for modules.

We shall prove

THEOREM 6.1. *Suppose that for any finitely generated faithful right A-module X and any finitely generated faithful right B-module Y the existence of a ring-isomorphism of $\mathrm{End}_A(X)$ onto $\mathrm{End}_B(Y)$ implies the existence of a category-isomorphism T from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ such that $T(X)$ is B-isomorphic to Y. If there exist some X and Y such that $\mathrm{End}_A(X)\cong\mathrm{End}_B(Y)$, and if A has a duality for modules, then A and B are quasi-Frobenius.*

Proof. We shall use the notations described at the beginning of §2. Thus $A^0=eAe$ and $B^0=e'Be'$ are self-basic rings of A and B respectively, and e and e' are idempotent elements of A and B respectively. Let $D=(D_1, D_2)$ be a duality between $_A\mathfrak{M}^*$ and $\mathfrak{M}_A^*$. We put $X_0=D_1(Ae)$, $Y_0=e'B$, where Ae is considered as a left A-module and Y is a right B-module. Then $\mathrm{End}_A(X_0)$ is inverse-isomorphic to the A-endomorphism ring of the left A-module Ae and the latter is inverse-isomorphic to A^0. Hence $\mathrm{End}_A(X_0)\cong A^0$. On the other hand, we have $\mathrm{End}_B(Y_0)\cong B^0$. Since there are some X and Y such that $\mathrm{End}_A(X)\cong\mathrm{End}_B(Y)$, A and B are similar by Morita [8, Theorem 7.5], and hence A^0 is ring-isomorphic to B^0. Therefore we have $\mathrm{End}_A(X_0)\cong\mathrm{End}_B(Y_0)$. Hence from the assumption of the theorem it follows that there is a category-isomorphism T from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ such that $T(X_0)$ is B-isomorphic to Y_0. Since the left A-module Ae is projective, X_0 is injective and hence $T(X_0)$ is also injective, and consequently Y_0 is injective. Therefore by a theorem of Ikeda (for example, cf. Morita [8, Theorem 14.1]) B is quasi-Frobenius, and A is also quasi-Frobenius. This completes our proof.

From Theorems 6.1 and 1.2 we obtain a characterization of quasi-Frobenius rings.

THEOREM 6.2. *A is a quasi-Frobenius ring if and only if A has a duality for modules and for any finitely generated faithful right A-modules X and Y the existence of a ring-isomorphism of $\mathrm{End}_A(X)$ onto $\mathrm{End}_A(Y)$ implies the existence of a category-isomorphism T from $\mathfrak{M}_A$ onto itself such that $T(X)$ is A-isomorphic to Y.*

([9]) A duality between $_A\mathfrak{M}^*$ and $\mathfrak{M}_A$ is defined to be a pair $D=(D_1, D_2)$ of contravariant functors $D_1: {_A\mathfrak{M}^*}\to\mathfrak{M}_A^*$ and $D_2: \mathfrak{M}_A^*\to{_A\mathfrak{M}^*}$ such that the composite functors D_2D_1 and D_1D_2 are naturally equivalent to the identity functor. Cf. Morita [8].

7. **The case of nonfaithful modules.** In this section we shall prove the following theorems.

THEOREM 7.1. *Let A be a uni-serial ring whose basic ring A^0 is decomposed into a direct sum of indecomposable two-sided ideals A_j^0 ($j = 1, \cdots, s$) such that if A_i^0/N_i^0 is ring-isomorphic to A_j^0/N_j^0 then A_i^0 is ring-isomorphic to A_j^0 where N_k^0 is the radical of A_k^0. Let B be similar to A. Then for a right A-module X and a right B-module Y the existence of a ring-isomorphism of $\mathrm{End}_A(X)$ onto $\mathrm{End}_B(Y)$ implies the existence of a category-isomorphism T from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ such that $T(X)$ is B-isomorphic to Y.*

THEOREM 7.2. *Suppose that for any finitely generated right A-module X and any finitely generated right B-module Y the existence of a ring-isomorphism of $\mathrm{End}_A(X)$ onto $\mathrm{End}_B(Y)$ implies the existence of a category-isomorphism T from $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ such that $T(X)$ is B-isomorphic to Y. If there exist some X and Y such that $\mathrm{End}_A(X) \cong \mathrm{End}_B(Y)$ and if every residue class ring of A has a duality for modules, then A is a uni-serial ring with the structure described in Theorem 7.1 and B is similar to A.*

THEOREM 7.3. *A is a uni-serial ring with the structure described in Theorem 7.1 if and only if every residue class ring of A has a duality for modules and for any finitely generated right A-modules X and Y the existence of a ring-isomorphism of $\mathrm{End}_A(X)$ onto $\mathrm{End}_A(Y)$ implies the existence of a category-isomorphism T from $\mathfrak{M}_A$ onto itself such that $T(X)$ is A-isomorphic to Y.*

Of course, Theorem 7.3 is a direct consequence of Theorems 7.1 and 7.2.

Proof of Theorem 7.2. We have only to consider the case where A and B are self-basic. In this case there is a ring-isomorphism θ of A onto B, since A is similar to B as is seen from the existence of a category-isomorphism of $\mathfrak{M}_A$ onto $\mathfrak{M}_B$ by [8, Theorem 7.5].

Let I be any two-sided ideal of A. Then A/I has a duality for modules and is self-basic. Since any finitely generated right A/I-module X is a finitely generated right A-module and $\mathrm{End}_A(X) \cong \mathrm{End}_{A/I}(X)$, it follows from Theorem 6.1 that A/I is quasi-Frobenius. Hence by a theorem of Ikeda [6] A is uni-serial.

Now let $A = A_1 \oplus \cdots \oplus A_s$ be a decomposition of A into a direct sum of indecomposable two-sided ideals of A. Let N_i be the radical of A_i. Suppose that A_i/N_i is ring-isomorphic to A_j/N_j.

We put $X = A_i/N_i$, $Y = B_j/M_j$, where $B_j = \theta(A_j)$, $M_j = \theta(N_j)$. Then $\mathrm{End}_A(X)$ and $\mathrm{End}_B(Y)$ are ring-isomorphic. Since A and B are self-basic, there exists a semi-linear isomorphism ω of X onto Y with a ring-isomorphism ϕ of A onto B. The correspondence $x \rightarrow \phi(x)(x \in A)$ induces a semi-linear isomorphism ω' of X onto the right B-module $\phi(A_i)/\phi(N_i)$ with a ring-isomorphism ϕ of A onto B, and hence the right B-modules Y and $\phi(A_i)/\phi(N_i)$ are B-isomorphic. Since a decomposition of B into a direct sum of indecomposable

two-sided ideals is unique, we see that $\phi(A_i) = B_j$, $\phi(N_i) = M_j$. Since $B_j = \theta(A_j)$, it follows that A_i are ring-isomorphic to A_j. This proves Theorem 7.2.

Before proving Theorem 7.1 we shall give a lemma which was proved by Asano [1] for the special case where X and Y are finitely generated.

LEMMA 7.4. *Let A be a primary uni-serial ring. Let X and Y be any two right A-modules. If $\mathrm{End}_A(X)$ is ring-isomorphic to $\mathrm{End}_A(Y)$, then X and Y are A-isomorphic.*

Proof. It is sufficient to prove the lemma for the case where A is self-basic, that is, A is a completely primary uni-serial ring. If we put $I = \{a \mid xa = 0 \text{ for all } x \in X\}$, $J = \{a \mid ya = 0 \text{ for all } y \in Y\}$, then X is a faithful right A/I-module and Y a faithful right A/J-module. Since A/I and A/J are quasi-Frobenius and self-basic, we see by Theorem 1.3 that there is a semi-linear isomorphism ω of X onto Y with a ring-isomorphism θ of A/I onto A/J. Since the totality of all two-sided ideals of A is $\{N^i \mid i = 0, 1, 2, \cdots\}$ where N is the radical of A, we have $I = J$.

Let $X = \sum \oplus X_\lambda$ be a decomposition of X into a direct sum of indecomposable right A/I-submodules. For each λ there exists some i such that $X_\lambda \cong A/N^i$, $I \subseteq N^i$. Then we have $\omega(X_\lambda) \cong A/N^i$ since $\theta(N^i/I) = N^i/I$. Thus $X_\lambda \cong \omega(X_\lambda)$ for each λ. Therefore X and Y are A/I-isomorphic, and consequently they are A-isomorphic. This proves Lemma 7.4.

Proof of Theorem 7.1. It is sufficient to prove the theorem for the case that A and B are self-basic. In this case there is a ring-isomorphism θ of A onto B since A and B are similar. Corresponding to the direct-sum decomposition of A into indecomposable two-sided ideals: $A = A_1 \oplus \cdots \oplus A_s$ we have a decomposition of B: $B = B_1 \oplus \cdots \oplus B_s$ where $B_i = \theta(A_i)$.

Let X be a right A-module and Y a right B-module. If we put $X_i = XA_i$, $Y_i = YB_i$, then we have the following decompositions:

$$X = X_1 \oplus \cdots \oplus X_s, \qquad Y = Y_1 \oplus \cdots \oplus Y_s,$$

$$\mathrm{End}_A(X) = \mathrm{End}_{A_1}(X_1) \oplus \cdots \oplus \mathrm{End}_{A_s}(X_s),$$

$$\mathrm{End}_B(Y) = \mathrm{End}_{B_1}(Y_1) \oplus \cdots \oplus \mathrm{End}_{B_s}(Y_s).$$

If we set $I_k = \mathrm{Ann}(X_k; A_k)$[10] and $J_k = \mathrm{Ann}(Y_k; B_k)$ then we have

$$\mathrm{End}_{A_k}(X_k) = \mathrm{End}_{A_k/I_k}(X_k) \quad \text{and} \quad \mathrm{End}_{B_k}(Y_k) = \mathrm{End}_{B_k/J_k}(Y_k).$$

Here we may assume without loss of generality that $X_k \neq 0$, $Y_k \neq 0$, $k = 1, \cdots, s$. Then the $\mathrm{End}_{A_k}(X_k)$-endomorphism ring of the left $\mathrm{End}_{A_k}(X_k)$-module X_k is indecomposable. Since X_k is faithful as a left $\mathrm{End}_{A_k}(X_k)$-module, the ring $\mathrm{End}_{A_k}(X_k)$ is indecomposable as a two-sided ideal of $\mathrm{End}_{A_k}(X_k)$. Consequently $\mathrm{End}_{A_k}(X_k)$ is indecomposable as a two-sided ideal of $\mathrm{End}_A(X)$. Likewise $\mathrm{End}_{B_k}(Y_k)$ is indecomposable as a two-sided ideal of $\mathrm{End}_B(Y)$.

---

[10] By $\mathrm{Ann}(X; A)$ we shall mean the annihilator ideal of A for a right A-module X.

Let Φ be a ring-isomorphism of $\mathrm{End}_A(X)$ onto $\mathrm{End}_B(Y)$. Since a decomposition of $\mathrm{End}_B(Y)$ into a direct sum of indecomposable two-sided ideals is unique (cf. Jacobson [7, p. 42]), there is a permutation π of $(1, 2, \cdots, s)$ such that $\Phi(\mathrm{End}_{A_i}(X_i)) = \mathrm{End}_{B_j}(Y_j)$ for $j = \pi(i)$.

Let $j = \pi(i)$.

Then X_i is a faithful right A_i/I_i-module, Y_j a faithful right B_j/J_j-module, and A_i/I_i, B_j/J_j are self-basic quasi-Frobenius. Hence by Theorem 1.3 there is a semi-linear isomorphism ω_i of the right A_i/I_i-module X_i onto the right B_j/J_j-module Y_j with a ring-isomorphism θ_i of A_i/I_i onto B_j/J_j. Let N_i be the radical of A_i. Then $I_i \subseteq N_i$,[11] and hence $\theta^{-1} \circ \theta_i$ induces a ring-isomorphism of A_i/N_i onto A_j/N_j. Therefore by the assumption on the structure of A it is seen that there is a ring-isomorphism ϕ_i of A_i onto A_j. If we put $\psi_i = \theta \circ \phi_i$, then ψ_i is a ring-isomorphism of A_i onto B_j. If we put further $y * a = y\psi_i(a)$ for $y \in Y_j$, $a \in A_i$, then Y_j is a right A_i-module and its endomorphism ring is isomorphic to $\mathrm{End}_{B_j}(Y_j)$. Hence the A_i-endomorphism rings of the right A_i-modules X_i and Y_j are isomorphic. By Lemma 7.4 we see that X_i and Y_j are A_i-isomorphic. Thus there exists a semi-linear isomorphism ω_i' of the right A_i-module X_i onto the right B_j-module Y_j with a ring-isomorphism ψ_i of A_i onto B_j. If we set

$$\psi\left(\sum_{i=1}^{s} a_i\right) = \sum_{i=1}^{s} \psi_i(a_i), \qquad\qquad a_i \in A_i,$$

$$\omega'\left(\sum_{i=1}^{s} x_i\right) = \sum_{i=1}^{s} \omega_i'(x_i), \qquad\qquad x_i \in X_i,$$

then ω' is a semi-linear isomorphism of the right A-module X onto the right B-module Y. This completes the proof of Theorem 7.1.

8. Modules over a uni-serial ring. Finally, we shall prove the following theorems which determine the general type of rings having the property described in Lemma 7.4.

Theorem 8.1. *If A is a uni-serial ring whose basic-ring A^0 is decomposed into a direct sum of indecomposable two-sided ideals $A_1^0, \cdots, A_s^0$ such that A_i^0/N_i^0 is not ring-isomorphic to A_j^0/N_j^0 for $i \neq j$ where N_k^0 denotes the radical of A_k^0, then for any two right A-modules X and Y the existence of a ring-isomorphism of $\mathrm{End}_A(X)$ onto $\mathrm{End}_A(Y)$ implies the existence of an A-isomorphism of X onto Y.*

Theorem 8.2. *The converse of Theorem 8.1 holds in case A has a duality for modules.*

Proof of Theorem 8.1. We have only to prove the case where A is self-basic. In this case, let $A = A_1 \oplus \cdots \oplus A_s$ be a decomposition of A into a direct sum of indecomposable two-sided ideals; each A_i is a completely

---

[11] In case $I_i = A_i$ we have $J_j = B_j$ and hence $X_i = 0$, $Y_j = 0$.

primary uni-serial ring and A_i/N_i is not ring-isomorphic to A_j/N_j for $i \neq j$ where N_k is the radical of A_k.

Let X and Y be right A-modules such that $\mathrm{End}_A(X)$ is ring-isomorphic to $\mathrm{End}_A(Y)$. If we set $X_i = XA_i$, $Y_i = YA_i$, $i = 1, \cdots, s$, then we have the following direct-sum decompositions:

$$X = \sum_{i=1}^{s} \oplus X_i, \qquad\qquad Y = \sum_{i=1}^{s} \oplus Y_i,$$

$$\mathrm{End}_A(X) = \sum_{i=1}^{s} \oplus \mathrm{End}_{A_i}(X_i), \qquad \mathrm{End}_A(Y) = \sum_{i=1}^{s} \oplus \mathrm{End}_{A_i}(Y_i).$$

Here we may assume without loss of generality that $X_i \neq 0$, $Y_i \neq 0$, $i = 1, \cdots, s$. Similarly as in the proof of Theorem 7.1 we see that there is a permutation π of $(1, \cdots, s)$ such that $\mathrm{End}_{A_i}(X_i) = \mathrm{End}_{A_j}(Y_j)$ for $j = \pi(i)$. If we denote by I_i resp. J_j the annihilators $\mathrm{Ann}(X_i; A_i)$ resp. $\mathrm{Ann}(Y_j; A_j)$, then A_i/I_i is ring-isomorphic to A_j/J_j. Since we have $I_i \subseteq N_i$, $J_j \subseteq N_j$[11], A_i/N_i is also ring-isomorphic to A_j/N_j. From the assumption on the structure of A it follows that $i = j$. Thus π is the identity. Hence by Lemma 7.4 $X_i \cong Y_i$ as right A_i-modules, and consequently X and Y are A-isomorphic.

Proof of Theorem 8.2. As in the proof of Theorem 7.2 we have only to prove the theorem for the case where A is self-basic.

From Theorem 6.2 it follows readily that A is quasi-Frobenius. Let N be the radical of A. Then by Nakayama [11, Part I, Theorem 6] we have $N^i = r(r(N^i)) = l(l(N^i))$, $i = 1, 2, \cdots$ where $r(S) = \{a \mid sa = 0 \text{ for every } s \in S\}$, $l(S) = \{a \mid as = 0 \text{ for every } s \in S\}$ for a subset S of A. Let X be any finitely generated left A-module such that $N^i X = 0$. Then we have $YN^i = 0$ for $Y = \mathrm{Char}_A X (= \mathrm{Hom}_A(X, A))$; because if $x \in X$, $\alpha \in Y$, $m \in N^i$ then $m\alpha(x) = \alpha(mx) = \alpha(0) = 0$ and hence $\alpha(x) \in r(N^i)$, and consequently $(\alpha m)(x) = \alpha(x)m = 0$ since $N^i = r(r(N^i))$. Similarly for any finitely generated right A-module Y with $YN^i = 0$ the left A-module $\mathrm{Char}_A Y$ is annihilated by N^i from the left-hand side. Therefore by associating with each finitely generated left or right A/N^i-module X the A-character module $\mathrm{Char}_A X$ we obtain a duality between the category of all finitely generated left A/N^i-modules and the category of all finitely generated right A/N^i-modules. Thus A/N^i has a duality for modules.

Now any finitely generated right A/N^i-module X is considered as a finitely generated right A-module, and $\mathrm{End}_A(X) = \mathrm{End}_{A/N^i}(X)$. Hence by Theorem 6.2 it is seen that A/N^i is quasi-Frobenius. In particular, A/N^2 is quasi-Frobenius. Hence by Nakayama [12] A is a generalized uni-serial ring. We shall prove furthermore that A is uni-serial. For this purpose by Nakayama [12] we have only to show that $B = A/N^2$ is uni-serial.

As is noted above B is self-basic, quasi-Frobenius and generalized uni-serial. Furthermore for any finitely generated right B-modules X and Y the

existence of a ring-isomorphism of $\mathrm{End}_B(X)$ onto $\mathrm{End}_B(Y)$ implies the existence of a B-isomorphism of X onto Y. Let e be a primitive idempotent element of B and M the radical of B. Then we have $\mathrm{End}_B(eB/eM)\cong eBe/eMe$. On the other hand, eM is the semi-simple part of the injective right B-module eB, and hence by our previous paper [10, Theorem 4.2] $\mathrm{End}_B(eM)\cong eBe/eMe$; because if ebe is an element of B such that $ebeeM = 0$ then $ebeeB$ is a simple right B-module and hence $ebeeB \subseteq eM$. Therefore eB/eM is B-isomorphic to eM. Consequently B is uni-serial. Thus A is uni-serial.

Let $A = A_1 \oplus \cdots \oplus A_s$ be a direct-sum decomposition of A into indecomposable two-sided ideals of A. Here it is assumed that A is self-basic. Moreover, since A is uni-serial as is shown above, every residue class ring of A has a duality for modules. Hence by Theorem 7.3 A has the structure described in Theorem 7.1; that is, if A_i/N_i is ring-isomorphic to A_j/N_j, then A_i is ring-isomorphic to A_j where N_k is the radical of A_k. Suppose that A_i/N_i is ring-isomorphic to A_j/N_j for $i \neq j$. Let us set $X = A_i/N_i$, $Y = A_j/N_j$; then X and Y are finitely generated right A-modules and $\mathrm{End}_A(X)$ is ring-isomorphic to $\mathrm{End}_A(Y)$, but X and Y are not A-isomorphic. Hence A_i/N_i is not ring-isomorphic to A_j/N_j for $i \neq j$. Thus the theorem is completely proved.

REFERENCES

1. K. Asano, *Über verallgemeinerte Abelsche Gruppen mit Operatorenring und ihre Anwendungen*, Jap. J. Math. 15 (1939), 231–253.

2. G. Azumaya, *Corrections and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem*, Nagoya Math. J. 1 (1950), 117–124.

3. ———, *A duality theory for injective modules*, Amer. J. Math. 81 (1959), 249–278.

4. R. Baer, *Automorphism rings of primary abelian operator groups*, Ann. of Math. 44 (1943), 192–227.

5. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, 1956.

6. S. Ikeda, *Some generalizations of quasi-Frobenius rings*, Osaka Math. J. 3 (1951), 227–238.

7. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ. Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.

8. K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku 6, No. 150 (1958), 83–142.

9. ———, *On algebras for which every faithful representation is its own second commutator*, Math. Z. 69 (1958), 429–434.

10. K. Morita, Y. Kawada and H. Tachikawa, *On injective modules*, Math. Z. 68 (1957), 217–226.

11. T. Nakayama, *On Frobeniusean algebras.* I, Ann. of Math. 40 (1939), 611–633; II, Ann. of Math. 42 (1941), 1–21.

12. ———, *Note on uni-serial and generalized uni-serial rings*, Proc. Imp. Acad. Tokyo 16 (1940), 285–289.

13. M. Osima, *Notes on basic rings.* II, Math. J. Okayama Univ. 3 (1954), 109–119.

14. K. Shoda, *Über die mit einer Matrix vertauschbaren Matrizen*, Math. Z. 29 (1929), 696–712.

TOKYO UNIVERSITY OF EDUCATION,
    TOKYO, JAPAN